Efficient estimation of the lower and upper bounds on the probability of failure of geotechnical models considering dependence and epistemic uncertainties

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ABSTRACT

Geotechnical engineering involves significant aleatory and epistemic uncertainties, often inadequately addressed in traditional reliability analyses, which tend to neglect epistemic uncertainties, simplify dependencies among random variables, and rely on less efficient methods such as Monte Carlo simulations, FORM, or SORM. This paper presents a reliability analysis method that integrates both aleatory and epistemic uncertainties using random sets, copulas for variable dependence, and the subset simulation algorithm for enhanced efficiency. Rather than providing a single-point probability of failure, our methodology computes bounds on failure probability whose width reflect the epistemic uncertainties in the input variables. The evaluated methodology allows geotechnical engineers to assess whether the bounds on the probability of failure are adequate or if more and better data are needed to refine them, leading to more reliable models. The applicability of the approach is demonstrated through three practical geotechnical examples.

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INTRODUCTION

Helton (1997) classifies uncertainties into aleatory (natural variability) and epistemic (lack of knowledge). In geotechnical engineering, aleatory uncertainty relates to the variability of materials, loads, and external agents, while epistemic uncertainty is linked to models, parameters, and site characterizations. Epistemic uncertainty can be reduced with better information, but aleatory uncertainty remains irreducible (Helton 1997).

Marek et al. (1996) and Hurtado (2004) categorize methodologies for quantifying structural safety under uncertainty into two groups: those using safety factors and those based on limit state functions. Safety factor-based methods, such as deterministic and semi-probabilistic analysis, are widely used due to their simplicity and broad application. In contrast, limit state function-based methodologies, like probabilistic analysis, estimate event probabilities by explicitly handling uncertainties.

Probabilistic analysis uses a limit state function $G : \mathcal{X} \to \mathbb{R}$ to divide the domain $\mathcal{X}$ into failure $F := \{\boldsymbol{X} \in \mathcal{X} : G(\boldsymbol{X}) \leq 0\}$ and safe $S := \mathcal{X} \setminus F$ regions, with the goal of estimating the probability of failure as follows:

$$P_f = \int \cdots \int_{\mathcal{X}} \mathbb{I}\left[G(\boldsymbol{X}) \leq 0\right] \mathrm{d}F_{\boldsymbol{X}}(\boldsymbol{x})\,; \tag{1}$$

here $F_{\boldsymbol{X}}(\boldsymbol{x})$ is the cumulative distribution function (CDF) of the input variables, and $\mathbb{I}[\cdot]$ stands for the *indicator function*, for which $\mathbb{I}[\cdot] = 1$ if the condition within the brackets is met, and $\mathbb{I}[\cdot] = 0$ otherwise. When $F_{\boldsymbol{X}}(\boldsymbol{x})$ is sufficiently differentiable, then the probability density function (PDF) $f_{\boldsymbol{X}}(\boldsymbol{x})$ exists, and equation (1) can be recast as

$$P_f = \int \cdots \int_{\mathcal{X}} \mathbb{I}\left[G(\boldsymbol{X}) \leq 0\right] f_{\boldsymbol{X}}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \tag{2}$$

Geotechnical reliability analysis, as described by equation (1), faces several challenges. First, equation (1) often lacks an analytical solution, and the failure domain usually lies in the tails of $F_{\boldsymbol{X}}(\boldsymbol{x})$, requiring computationally expensive numerical approximations for precision. Second, due to limited data, input variable marginal CDFs are often imprecise, and relationships between variables are poorly defined, with independence assumed without basis. Thirdly, oftentimes the information of the input variables cannot be expressed in terms of CDFs but in other structures of uncertainty such as probability boxes or possibility distributions; sometimes authors try to model epistemic information as CDFs; however this approach involves a loss of information (see, e.g., Beer et al. 2013). Lastly, equation (1) is typically solved using a probabilistic approach, which is effective for aleatory uncertainties but inadequate for epistemic uncertainties.

The limitations of conventional probabilistic analysis have prompted research into alternative approaches. This study utilizes three well-established and widely validated methodologies: subset simulation, copulas, and random sets theory. Subset simulation enables efficient and accurate estimation of small failure probabilities through random walk exploration of the failure domain using Markov chains (e.g., Au and Beck 2001). Copula theory is key to modeling the dependence between random variables (e.g., Nelsen 2007). Finally, random sets theory provides a unified framework for handling both aleatory and epistemic uncertainties, expressed through various uncertainty structures (e.g., Bernardini and Tonon 2010). Further details of these theories will be provided in subsequent sections.

While each of these three theories has been applied individually in geotechnical analyses, they are often used in isolation without integration into a unified methodology. However, Alvarez et al. (2018) proposed a framework that combines subset simulation, copula theory, and random sets, which will be validated in this work for geotechnical reliability analysis.

This article is organized as follows: first, a brief overview of the subset simulation algorithm is provided; the next two sections cover the basics of copula functions and random sets theory; then, the integration of these three theories into a comprehensive methodology for geotechnical reliability analysis is explained; the methodology is validated through three practical examples; and finally, the article concludes with a discussion and conclusion section summarizing the key findings.

SUBSET SIMULATION FOR ASSESSING RELIABILITY IN GEOTECHNICS

Subset simulation (SubSim) (Au et al. 2007) is an algorithm for computing small probabilities of failure. In this approach, the original problem of computing a small probability of failure is expressed as the product of larger values of conditional probabilities of intermediate failure events b_j. In other words, the original problem related to a rare event is transformed into a sequence of simulations involving more frequent events.

Let F denote the failure event, and let $\mathscr{X} \supset F_1 \supset F_2 \supset \cdots \supset F_m = F$ be a decreasing nested sequence of failure events, such that $F_k = \bigcap_{j=1}^{k} F_j$ for $k = 1, 2, \ldots, m$. By definition of conditional probability, the probability of F can be expressed as (see, e.g., Au and Beck 2001):

$$P_f = P(F_m) = P(F_1) \prod_{j=2}^{m} P\left(F_j | F_{j-1}\right) \tag{3}$$

Even if the probability of failure is small, by appropriately selecting the intermediate failure events $F_j : j = 2, 3, \ldots, m$, the conditional probabilities in equation (3) can be made large enough to facilitate efficient simulation. This transforms the problem of simulating a rare event into a series of more frequent events in the conditional probability space, which are quicker to simulate.

The intermediate failure events $F_j : j = 1, \ldots, m$ depend on two key aspects: the parametrization of the failure domain F and the sequence of intermediate failure domains. The parametrization involves defining the failure event using a performance function G, with the failure domain as $F = \{\boldsymbol{x} \in \mathscr{X} : G(\boldsymbol{x}) > b\}$, and the sequence of intermediate events as $F_j = \left\{\boldsymbol{x} \in \mathscr{X} : G(\boldsymbol{x}) > b_j\right\}$, where $b_1 > b_2 > \cdots > b_m = b$.

Choosing the sequence of thresholds $\left\{b_j : j = 1, \ldots, m\right\}$ can be challenging, but adaptive values of b_j based on a prescribed conditional probability $p_0 \in (0, 1)$ can be used. Here, b_j is defined as the $[(1 - p_0) N]$-th largest value among the samples $\left\{G\left(\mathbf{x}_j^{(i)}\right) : i = 1, \ldots, N\right\}$ obtained at each conditional level. This adaptive approach ties the thresholds b_j to p_0 and sample values, making them vary across different simulations.

Zuev et al. (2012) showed that setting $p_0 \in [0.1, 0.3]$ gives similar efficiency to the optimal value of p_0, eliminating the need for further refinement. Additionally, it is recommended to choose p_0 so that $p_0 N$ and $(1 - p_0) N$ are positive integers, although this is not mandatory. Note that the estimated conditional probability $P\left(F_j | F_{j-1}\right)$ may not exactly equal p_0, depending on the sample size N. Thus, N must be large enough to ensure the approximation $P\left(F_j | F_{j-1}\right) \approx p_0$ is accurate.

To estimate P_f using equation (3), it's necessary to compute $P(F_1)$ and $P\left(F_j | F_{j-1}\right)$ for $j = 2, 3, \ldots, m$. The first can be easily estimated using crude MCS as:

$$P(F_1) \approx \frac{1}{N} \sum_{k=1}^{N} \mathbb{I}\left[\boldsymbol{x}_k \in F_1\right] \tag{4}$$

The conditional probabilities $P\left(F_j | F_{j-1}\right)$ are computed based on a estimator similar to equation (4):

$$P\left(F_j | F_{j-1}\right) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}\left[\boldsymbol{x}_{j-1}^{(i)} \in F_j\right]. \tag{5}$$

Samples must be simulated according to the conditional distribution $\pi\left(\cdot | F_{j-1}\right)$. This is efficiently done using a customized Metropolis-Hastings algorithm (see Au and Beck 2001), which generates $(1 - p_0) N$ additional samples based on the initial $p_0 N$ samples in F_1. These are used to estimate $P(F_2 | F_1)$ by defining F_2 with a prescribed conditional probability p_0. The process is repeated, generating new samples according to $\pi\left(\cdot | F_j\right)$ to estimate $P(F_{j+1} | F_j)$, until F_m is reached.

Finally, combining equations (4) and (5), the probability of failure is given by equation (3). The SubSim algorithm is more efficient than crude MCS, particularly for small failure probabilities (Au and Beck 2003). More details can be found in (Au and Beck 2001) and (Zuev et al. 2012).

DEPENDENCE MODELLING THROUGH COPULA THEORY

A copula $\mathtt{C}$ is a d-dimensional CDF whose domain is $[0, 1]^d$ and whose range is $[0, 1]$, that is $\mathtt{C} : [0, 1]^d \to [0, 1]$. In this regard, every marginal of a d-dimensional copula is uniform in the interval $[0, 1]$.

The term "copula" was introduced by Sklar (1959) in probability theory to describe functions that link marginal CDFs through a dependence structure to form joint CDFs, capturing all dependence information of a joint probability distribution. This is established by *Sklar's theorem.*

Let us consider a multivariate distribution function $F_{X_1, X_2, \ldots X_d}$ with marginal distributions F_{X_i} for $i = 1, \ldots, d$. Sklar's theorem dictates that there exists a copula $\mathtt{C}$ such that:

$$F_{X_1, X_2, \ldots X_d}(\boldsymbol{x}) = \mathtt{C}\left(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_d}(x_d)\right) \tag{6}$$

for all $\boldsymbol{x} := [x_1, x_2, \ldots x_d] \in \mathbb{R}^d$. This copula is *unique* if all marginal distributions F_{X_i} are continuous; otherwise, the copula $\mathtt{C}$ is uniquely defined on the Cartesian product of the range of each marginal distribution, that is range $(F_1) \times$ range $(F_2) \times \cdots \times$ range (F_d). Analogously, by knowing a copula $\mathtt{C}$ and a group of one dimensional CDFs F_{X_i}, the d-dimensional distribution function $F_{X_1,X_2,\ldots,X_d}$ can be defined by the equation (6).

For details on practical concepts and the applicability of copula theory in geotechnical engineering, readers are referred to the tutorial and state-of-the-art review presented by Sepulveda-Garcia and Alvarez (2022).

BASIC CONCEPTS OF RANDOM SETS

Random sets theory provides a framework to handle both aleatory and epistemic uncertainties without assuming or converting epistemic uncertainty into aleatory uncertainty, avoiding the loss of information (see Beer et al. 2013). Random sets theory originates from Kendall (1974) and Matheron (1974), who studied random variables in closed bounded sets of $\mathbb{R}^d$. This theory extends point-valued probability to set-valued probability, integrating uncertainty structures such as intervals, Dempster-Shafer theory, possibility distributions, probability boxes, and CDFs.

This section introduces random sets theory and its role in geotechnical reliability analysis, addressing both aleatory and epistemic uncertainties. It begins with an explanation of *Dempster-Shafer evidence theory*, followed by an introduction to random sets and their relation to other uncertainty representation structures.

Dempster-Shafer evidence theory

Dempster-Shafer evidence theory (DST) is a mathematical framework for representing aleatory and epistemic uncertainties. It was introduced by Dempster (1967) and expanded by Shafer (1976). DST extends conventional probability by assigning probabilities to sets instead of singletons (Sentz et al. 2002).

According to Dubois and Prade (1991), let $\mathscr{X}$ be a non-empty set and $\mathscr{P}(\mathscr{X})$ its power set. Let $\mathscr{F} := \{A_1, A_2, \ldots, A_n\}$ be a family of non-empty, distinct subsets of $\mathscr{X}$, such that $A_i \in \mathscr{P}(\mathscr{X}) \setminus \emptyset$ and $A_i \neq A_j$ for all $i, j = 1, 2, \ldots, n$ and $i \neq j$. Define $m : \mathscr{F} \longrightarrow [0,1]$ as a mapping from $\mathscr{F}$ to the unit interval, where each A_i is a focal element with an associated basic mass assignment $m(A_i) \geq 0$. The collection of these elements, $\mathscr{F}$, is the focal set, and the pair $(\mathscr{F}, m)$ forms a Dempster-Shafer body of evidence or a Dempster-Shafer structure. Note that every A_i contains the possible values of the variable $\boldsymbol{x} \in \mathscr{X}$ and $m(A_i)$ is the probability of A_i being the true range of $\boldsymbol{x}$. Here, $m(A_i)$ represents the probability that A_i is the true range of $\boldsymbol{x}$, with $m(\emptyset) = 0$ and $\sum_{A \in \mathscr{F}} m(A) = 1$.

The probability mass assignment of A_i indicates how much the information supports that an element of $\mathscr{X}$ belongs to A_i, but it does not specify whether the element is within subsets of A_i (Klir 1995). Any further details about the element's position within A_i must be represented by another focal element and its mass assignment (Bernardini and Tonon 2010).

Dempster-Shafer theory asserts that, with imprecise information, the probability of an element $\boldsymbol{x} \in F$ (where $F \in \mathscr{F}$) is bounded by two thresholds: $\mathrm{Bel}_{(\mathscr{F},m)}(F) \leq P(F) \leq \mathrm{Pl}_{(\mathscr{F},m)}(F)$. Here, $\mathrm{Bel}_{(\mathscr{F},m)}(F)$ is the *belief function*, and $\mathrm{Pl}_{(\mathscr{F},m)}(F)$ is the *plausibility function*, defined as:

$$\begin{aligned} \mathrm{Bel}_{(\mathscr{F},m)}(F) &:= \sum_{i=1}^{n} \mathbb{I}[A_i \subseteq F]\, m(A_i) \\ \mathrm{Pl}_{(\mathscr{F},m)}(F) &:= \sum_{i=1}^{n} \mathbb{I}[A_i \cap F \neq \emptyset]\, m(A_i). \end{aligned} \tag{7}$$

The belief measure of F is the sum of basic mass assignments for focal elements within F that imply $\boldsymbol{x} \in F$. The plausibility measure is the sum of mass assignments for focal elements that overlap with F and may imply $\boldsymbol{x} \in F$. If $\mathrm{Bel}_{(\mathscr{F},m)}(A_i) = \mathrm{Pl}_{(\mathscr{F},m)}(A_i)$ for all $i = 1, \ldots, n$, DS theory reduces to conventional probability theory.

Succinct introduction to random sets

Consider a non-empty set $\mathscr{X}$ and its power set $\mathscr{P}(\mathscr{X})$. Let (Ω, σ_Ω) be a probability space and $(\mathscr{F}, \sigma_{\mathscr{F}})$ a measurable space. A random set (RS) Γ is a set-valued random variable $\Gamma : \Omega \to \mathscr{F}$ that generates a probability measure on $(\mathscr{F}, \sigma_{\mathscr{F}})$, such that $P_\Gamma := P_\Omega \circ \Gamma^{-1}$. Therefore, an event F has the probability:

$$P_\Gamma(F) = P_\Omega(\{\omega \in \Omega : \Gamma(\omega) \in F\}) \tag{8}$$

if all elements of $\mathscr{F}$ are singletons (no epistemic uncertainty), the random set becomes a random variable, i.e., $\Gamma(\omega) = X(\omega)$.

Random sets theory asserts that with epistemic uncertainty, the exact value of $P_X(F)$ is unknown, but its lower and upper probability bounds are defined as (Dempster 1967):

$$\begin{aligned} \underline{P_\Gamma}(F) &:= P_\Omega(\{\omega : \Gamma(\omega) \subseteq F, \Gamma(\omega) \neq \emptyset\}) \\ \overline{P_\Gamma}(F) &:= P_\Omega(\{\omega : \Gamma(\omega) \cap F \neq \emptyset\}); \end{aligned} \tag{9}$$

in this sense, $P_X(F)$ is bounded by the inequality $\underline{P_\Gamma}(F) \leq P_X(F) \leq \overline{P_\Gamma}(F)$.

Other representations of uncertainty and their relationship with random sets

Alvarez (2006) proved that a the general definition of a random set can be particularized in the probability space as follows:

- $\Omega := (0,1]^d$.
- $\sigma_\Omega := \mathscr{B}(\Omega)$, where $\mathscr{B}(\Omega)$ is the Borel σ-algebra on Ω.
- $P_\Omega := \mu_{\mathtt{C}}$ for some copula function $\mathtt{C}$ which contains the dependence information within the joint random set; here $\mu_{\mathtt{C}}$ denotes the Lebesgue-Stieltjes measure associated to the copula $\mathtt{C}$; in other words, it is probability measure of a given region under the copula $\mathtt{C}$.
- d-boxes as elements of $\mathscr{F}$.

This allows the use of other uncertainty representations in random set theory, such as probability distributions, interval families, possibility distributions, and probability boxes, which will be introduced below.

Probability distribution functions

Consider a random variable X with a continuous CDF F_X. The realization $F_X(x)$ is uniformly distributed in $(0,1]$. If $\mathscr{F} = \left\{x : x := F_X^{(-1)}(\alpha), \alpha \in (0,1]\right\}$, then X is associated with a random set $(\mathscr{F}, P_\Gamma)$, where $\Gamma(\alpha) = F_X^{(-1)}(\alpha)$ and $P_\Gamma = \alpha$. If α has an associated x_i, then $x_1 \le x_2$ if $\alpha_1 \le \alpha_2$.

Intervals

An interval $I = [a, b]$ represents values between a lower limit a and an upper limit b, without specifying the CDF or shape between them. It should not be confused with a uniform CDF on $[a, b]$, as the interval expresses a lack of knowledge, while the uniform CDF represents randomness.

To index a finite family of intervals $(\mathscr{F}, m')$ by α, a reproducible and unique ordering is needed. If $\{[a_i, b_i]$, for $i = 1, \dots, s\}$ are the focal elements of $\mathscr{F}$, then intervals can be lexicographically sorted by $[a_i, b_i] \le [a_j, b_j]$ if $a_i \le a_j$ or $b_i \le b_j$ when $a_i = a_j$. Sorting can be done using standard sorting algorithms like quicksort. More details can be found in Alvarez (2007).

Thus, a finite random set $(\mathscr{F}_s, m')$ is represented by $\Gamma(\alpha) := A^*(\alpha)$ for $\alpha \in (0, 1]$, where $A^*(\alpha)$ is the focal element satisfying $\sum_{k=i_1}^{i_{j-1}} m'(A_k) < \alpha \le \sum_{k=1}^{i_j} m'(A_k)$. By convention, $\sum_{k=i_1}^{0} m'(A_k) = 0$.

Possibility distributions

A possibility distribution is a mapping $A : X \to [0, 1]$, where $\sup_{x \in X} A(x) = 1$, representing the degree to which the concept in A fits x. Let A be a possibility distribution with membership function $A(x)$ on $X \in \mathbb{R}$, and $\mathscr{F}$ be the family of α-cuts A_α, i.e., $\mathscr{F} := \{\Gamma(\alpha) := A_\alpha : \alpha \in (0, 1]\}$. The possibility distribution can then be represented by a RS $(\mathscr{F}, P_\Gamma)$, where $\Gamma(\alpha) = A_\alpha$ and $P_\Gamma : \sigma_{\mathscr{F}} \to [0, 1]$ is defined as in equation (8).

Probability boxes

Consider two non-decreasing functions, $\overline{F}$ and $\underline{F}$, mapping $\mathbb{R} \to [0, 1]$, where $\underline{F}(x) \le \overline{F}(x)$ for all $x \in \mathbb{R}$. The set of all non-decreasing functions F such that $\underline{F}(x) \le \overline{F}(x)$ is called a *probability box* or *p-box*. P-boxes can be classified into two categories: distributional p-boxes, where the shape of the distribution is known but its parameters are uncertain (specified by intervals), and distribution-free p-boxes, where the distribution itself is unknown, and the bounds $\overline{F}$ and $\underline{F}$ are explicitly provided without assuming a specific parental distribution (Crespo et al. 2013).

Let us consider the distribution-free probability box $\langle \underline{F}, \overline{F} \rangle$ defined on a subset of $\mathbb{R}$. Based on it, let us define the focal set $\mathscr{F} = \left\{\Gamma(\alpha) := \langle \underline{F}, \overline{F} \rangle^{-1}(\alpha) := \left[\overline{F}^{(-1)}(\alpha), \underline{F}^{(-1)}(\alpha)\right] : \alpha \in (0, 1]\right\}$ where $\underline{F}^{(-1)}(\alpha)$ and $\overline{F}^{(-1)}(\alpha)$ are the quasi-inverses of $\underline{F}$ and $\overline{F}$, respectively. In this way, the probability box $\langle \underline{F}, \overline{F} \rangle$ can be represented by a random set $(\mathscr{F}, P_\Gamma)$, such that $\Gamma(\alpha) = \langle \underline{F}, \overline{F} \rangle^{-1}(\alpha)$ and P_Γ is obtained through equation (8).

Various methods exist for constructing distribution-free p-boxes from both empirical and theoretical data (Ferson et al. 2003). For example the Kolmogorov-Smirnov (K-S) confidence limits can be used when only a small sample is available (see Kolmogoroff 1941; Smirnov 1939). In contrast, distributional p-boxes require transformation into distribution-free p-boxes when used in random set theory, resulting in a loss of information due to the inability to specify the parental CDF (Alvarez et al. 2017).

PROPOSED APPROACH FOR THE ESTIMATION OF THE BOUNDS ON P_F IN GEOTECHNICAL ENGINEERING

This section demonstrates how subset simulation, copula theory, and random sets theory can be integrated into a unified methodology for reliability analysis, addressing both aleatory and epistemic uncertainties, along with dependencies between model variables. The approach is primarily based on the work of Alvarez and Hurtado (2014) and Alvarez et al. (2018).

Theoretical basis of the procedure

Using random sets theory, the precise probability of failure $P_X(F)$ for a geotechnical model cannot be computed. Instead, only its lower and upper bounds can be determined, as shown in equations (9). This is due to the lack of complete certainty about the exact multivariate CDF of the input variables. To evaluate these bounds, an alternative representation is needed. As discussed in section 4, using $\Omega := (0, 1]^d$, $\sigma_\Omega := \mathscr{B}(\Omega)$, with copula function $\mathtt{C}$ containing the dependence information and d-boxes as elements of $\mathscr{F}$, enables modeling of possibility distributions, probability boxes, Dempster-Shafer structures, and probability functions in a unified framework.

Using a method similar to the inverse transform in probability theory or α-cut sampling in fuzzy sets, we relate a value of α_i to an interval $\Gamma_j(\alpha_j)$, as shown in Figure 1. These intervals form a focal element, $\Gamma(\boldsymbol{\alpha}) = \bigtimes_{i=1}^{d} \Gamma_i(\alpha_i)$. This approach represents focal elements in $\mathscr{X}$ as points in Ω. Each focal element in $\mathscr{X}$ corresponds to a d-dimensional box, while in Ω, it is a point $\boldsymbol{\alpha} := [\alpha_1, \dots, \alpha_i, \dots, \alpha_d]$. Ω also captures the interdependence of the basic variables using copulas, with $\boldsymbol{\alpha}$ interpreted as a point sampled from the copula $\mathtt{C} : \Omega \to [0, 1]$. Figure 2 illustrates this in a bidimensional case.

Bounding the probabilities of failure with random sets

Given the probability space and a failure region F in $\mathscr{X}$, the sample set Ω contains two subregions: $F_{LP} := \{\boldsymbol{\alpha} \in \Omega : \Gamma(\boldsymbol{\alpha}) \subseteq F, \Gamma(\boldsymbol{\alpha}) \neq \emptyset\}$, where $\Gamma(\boldsymbol{\alpha})$ is fully contained in F, and $F_{UP} := \{\boldsymbol{\alpha} \in \Omega : \Gamma(\boldsymbol{\alpha}) \cap F \neq \emptyset\}$, where $\Gamma(\boldsymbol{\alpha})$ has at least one point in F. These sets are called the lower and upper inverses of F, respectively. Note that $F_{LP} \subseteq F_{UP}$, and equations (9) can be computed using Lebesgue-Stieltjes integrals (Alvarez 2006; Alvarez 2009).

$$
\begin{aligned}
\underline{P_\Gamma}(F) &= \int_\Omega \mathbb{I}[\boldsymbol{\alpha} \in F_{LP}]\, \mathrm{d}\mathtt{C}(\boldsymbol{\alpha}) = \mu_{\mathtt{C}}(F_{LP}) \\
\overline{P_\Gamma}(F) &= \int_\Omega \mathbb{I}[\boldsymbol{\alpha} \in F_{UP}]\, \mathrm{d}\mathtt{C}(\boldsymbol{\alpha}) = \mu_{\mathtt{C}}(F_{UP}).
\end{aligned} \tag{10}
$$

Since F_{LP} and F_{UP} are $\mu_{\mathtt{C}}$-measurable sets, the lower and upper probabilities of failure are given by $\underline{P_f} = \underline{P_\Gamma}(F)$ and $\overline{P_f} = \overline{P_\Gamma}(F)$. Additionally, Alvarez et al. (2018) states that two limit state functions, $\underline{G}$ and $\overline{G}$, can be defined in Ω such that the following relations hold:

$$
\begin{aligned}
\mathbb{I}[\Gamma(\boldsymbol{\alpha}) \subseteq F] = \mathbb{I}[\boldsymbol{\alpha} \in F_{LP}] = \mathbb{I}\left[\overline{G}(\boldsymbol{\alpha}) \le 0\right] \\
\mathbb{I}[\Gamma(\boldsymbol{\alpha}) \cap F \neq \emptyset] = \mathbb{I}[\boldsymbol{\alpha} \in F_{UP}] = \mathbb{I}\left[\underline{G}(\boldsymbol{\alpha}) \le 0\right].
\end{aligned} \tag{11}
$$

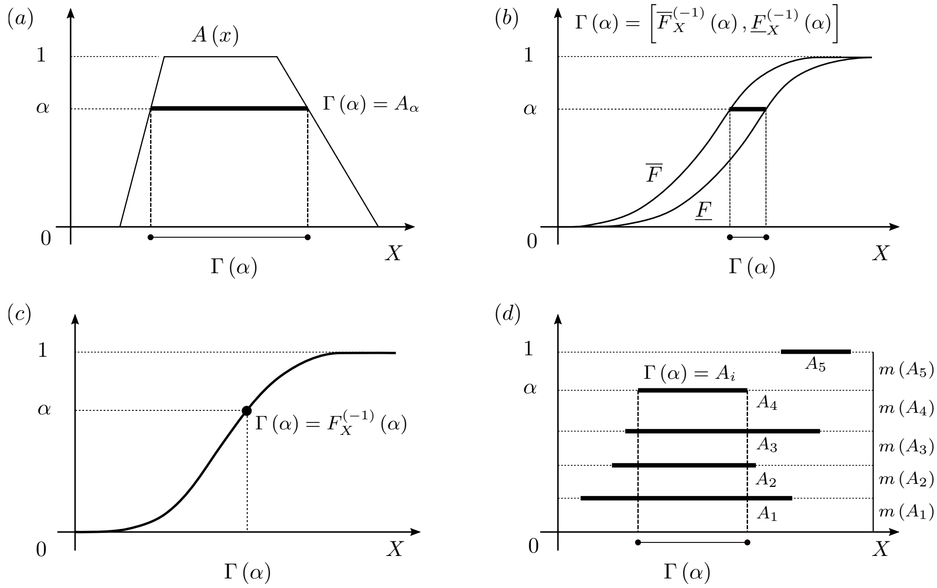

Fig. 1. Sampling of focal elements. (a) from a possibility distribution A, (b) from a probability box $\langle \underline{F}, \overline{F} \rangle$, (c) from a CDF F_X, and (d) from a Dempster-Shafer structure $(\mathscr{F}, m)$ (adapted from Alvarez (2007)).

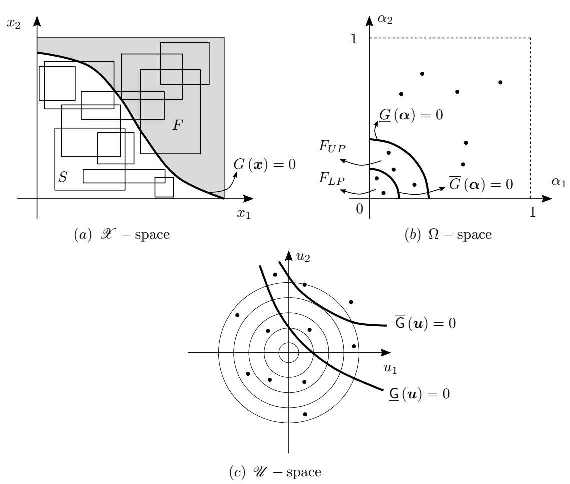

Fig. 2. The three sets used to solve the interval reliability problem. (a) In $\mathscr{X}$, the space of basic variables, focal elements are depicted as d-dimensional boxes; (b) In Ω, the dependence of the basic variables is defined by the copula $\mathtt{C}$, and where each point $\boldsymbol{\alpha}$ depicts a focal element; (c) the standard Gaussian probability space $\mathscr{U}$, where the SubSim algorithm is carried out (adapted from Alvarez and Hurtado (2014)).

Therefore, the bounds of the probability of failure can be expressed as (Alvarez and Hurtado 2014):

$$\begin{aligned} \underline{P_f} &= \int_{\Omega} \mathbb{I}\left[\overline{G}(\boldsymbol{\alpha}) \leq 0\right] \mathrm{d}\mathtt{C}(\boldsymbol{\alpha}) \\ \overline{P_f} &= \int_{\Omega} \mathbb{I}\left[\underline{G}(\boldsymbol{\alpha}) \leq 0\right] \mathrm{d}\mathtt{C}(\boldsymbol{\alpha}); \end{aligned} \tag{12}$$

these equations can be computed using the consistent and unbiased estimators, provided by Monte Carlo sampling:

$$\begin{aligned} \underline{\hat{P}_f} &= \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}\left[\boldsymbol{\alpha}_i \in F_{LP}\right] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}\left[\overline{G}(\boldsymbol{\alpha}_i) \leq 0\right] \\ \overline{\hat{P}_f} &= \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}\left[\boldsymbol{\alpha}_i \in F_{UP}\right] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}\left[\underline{G}(\boldsymbol{\alpha}_i) \leq 0\right] \end{aligned} \tag{13}$$

where the sample $\boldsymbol{\alpha}_i$ is drawn from the d-dimensional copula $\mathtt{C}$. Several algorithms in the literature can be used to obtain samples from a given copula (see, e.g., Nelsen 2007; Embrechts et al. 2001).

According to (11), $\Gamma(\boldsymbol{\alpha}) \subseteq F$ implies that $\Gamma(\boldsymbol{\alpha}) \cap F \neq \emptyset$. Hence, $\overline{G}(\boldsymbol{\alpha}) \leq 0$ leads to $\underline{G}(\boldsymbol{\alpha}) \leq 0$. By considering this principle, the number of evaluations of the function $\overline{G}$ can be significantly reduced.

Integration of Subset simulation in the estimation of probability bounds

The estimation of integrals in (12) can be performed with any algorithm for failure probability estimation. However, many algorithms, including subset simulation, operate in the standard Gaussian space $\mathscr{U}$. Therefore, a transformation between Ω and $\mathscr{U}$ is needed. This can be achieved through a bijective function T that maps $\boldsymbol{\alpha} \in \Omega$ (defined by the copula $\mathtt{C}$) to an independent point in $\mathscr{U}$, as explained in Lemaire (2013, chapter 4) and Ditlevsen and Madsen (1996, chapter 7). Defining $\boldsymbol{\alpha} = T^{-1}(\boldsymbol{\Phi}(\boldsymbol{u}))$, equations (12) become:

$$\underline{P_f} = \int_{\mathscr{U}} \mathbb{I}\left[\overline{\mathsf{G}}(\boldsymbol{u}) \leq 0\right] \mathrm{d}\Phi_U(\boldsymbol{u}) \tag{14}$$

$$\overline{P_f} = \int_{\mathscr{U}} \mathbb{I}\left[\underline{\mathsf{G}}(\boldsymbol{u}) \leq 0\right] \mathrm{d}\Phi_U(\boldsymbol{u}) \tag{15}$$

The limit state functions $\underline{\mathsf{G}}$ and $\overline{\mathsf{G}}$ are defined as $\underline{\mathsf{G}}(\boldsymbol{u}) = \underline{G}\left(T^{-1}(\boldsymbol{\Phi}(\boldsymbol{u}))\right)$ and $\overline{\mathsf{G}}(\boldsymbol{u}) = \overline{G}\left(T^{-1}(\boldsymbol{\Phi}(\boldsymbol{u}))\right)$, where $\boldsymbol{\Phi}(\boldsymbol{u}) =$

$[\Phi(u_1), \ldots \Phi(u_d)]$ is the joint standard Gaussian CDF. These functions are illustrated in figure 2c, where concentric circles represent the bivariate standard Gaussian distribution. Each point in $\mathcal{U}$ (figure 2c) corresponds to a point in Ω (figure 2b), which in turn maps to a d-dimensional box in $\mathcal{X}$ (figure 2a).

NUMERICAL EXAMPLES

This section illustrates the proposed approach with three examples that integrate both aleatory and epistemic uncertainties, considering the dependence between input variables. The SubSim algorithm ensures efficient simulation.

In all the examples, SubSim uses a conditional probability of $p_0 = 0.1$ for each intermediate level, generating $N = 1000$ focal elements per level. A uniform proposal distribution with a width of 2 units is applied. The image of each focal element $\Gamma(\boldsymbol{\alpha})$ through the limit state function G is determined using the sampling method from Alvarez (2007), which draws 100 samples from the focal element's domain.

Rotational failure of an homogeneous soil slope

Consider a finite homogeneous slope with a rotational failure mechanism, as shown in figure 3. The soil's mechanical behavior follows the Mohr-Coulomb failure criterion, with cohesion c, friction angle ϕ, and unit weight γ.

This example considers the values of ϕ presented by Oberguggenberger and Fellin (2002) with cohesion c set to 0 kPa. The friction angle ϕ is represented by a distributional free-P-box with an 80% confidence interval (see Ferson et al. 2003), and the unit weight γ follows a distributional p-box $\mathcal{N}([18, 22], [1, 2])$ kN/m^3.

The Kendall's tau correlation coefficient between the friction angle and unit weight is +0.1, and their dependence is modeled by a Nelsen No. 16 copula. Cohesion is independent of ϕ and γ. For the simulation, the unit weight distributional p-box must be transformed into a distributional-free p-box using the method in Figure 1b.

Using this information, the bounds $\underline{P_f}$ and $\overline{P_f}$ of the slope's probability of failure are computed with the Fellenius method. The limit state function is expressed as:

$$G(c, \phi) = FS - 1 \tag{16}$$

where FS is computed using the so-called Fellenius formula, as (see, e.g. Fredlund and Krahn (1977)):

$$FS = \frac{\sum_{i=1}^{n} \left[c_i L_i + (W_i \cos(\psi_i) - K_h W_i \sin(\psi_i) - U_i) \tan(\phi_i)\right]}{\sum_{i=1}^{n} \left[W_i \sin(\psi_i) + K_h W_i \cos(\psi_i)\right]}. \tag{17}$$

The Fellenius method divides the slope into n slices, $i = 1, \ldots, n$, each analyzed through limit equilibrium. In equation (17), i refers to a slice. Parameters are defined as follows: c and ϕ are the cohesion and friction angle at the slice base, L is the slice base length, W is the slice weight, U is the water pressure at the base, ψ is the angle between the vertical and the line from the slice's base to the failure circle center, and K_h is the seismic acceleration coefficient (set to 0.00 here). The model uses 11 slices of width $b_i = 29/11$ m.

With these parameters, equations (14) and (15) are computed using the method presented in section 5, yielding probability bounds $\left[\underline{P_f}, \overline{P_f}\right] = \left[7.58 \times 10^{-4}, 0.253\right]$.

This two-dimensional model illustrates how samples are distributed across three spaces: the original space of basic variables ($\mathcal{X}$-space), the copula domain (Ω-space), and the standard Gaussian domain ($\mathcal{U}$-space), where SubSim solves the reliability problem. Figure 4 shows the distribution and propagation of SubSim samples in these spaces, along with the failure surface $G(c, \phi)$ in the $\mathcal{X}$-space. Notably, calculating the upper probability of failure $\overline{P_f}$ did not require intermediate failure levels, while the lower probability $\underline{P_f}$ involved three. The figure also shows how SubSim redirects the simulation to the failure domain, improving efficiency in calculating failure probability bounds.

Plane failure of a rock slope

This example involves excavating a rock mass for a new highway, creating a 20-meter-high slope ($H = 20$ m) at a face angle of $\psi_f = 60°$. The rock mass has persistent bedding planes dipping at $\psi_p = 35°$ into the excavation. Behind the crest, a tension crack of depth Z will be filled with water to a height of Z_w above the sliding surface.

The rock mass follows a Mohr-Coulomb failure criterion, with failure modeled by cohesion c, friction angle ϕ, and unit weight γ. The impact of an earthquake generating a horizontal acceleration of $K_h g$ is considered, where K_h is the seismic acceleration coefficient and g is gravity. To counteract this load, active horizontal anchors are proposed to transmit a load T to the wedge.

Figure 5 shows the rock slope scheme, where U and V represent the resultant forces from hydrostatic pressures in the sliding plane and tension crack, respectively. The anchor is horizontal, so $\psi_T = 0$.

In a deterministic approach, the factor of safety (FS) of the slope can be formulated as follows (see, e.g., Hoek and Bray 1981; Wyllie 2017):

$$FS = \frac{cA + N' \tan(\phi)}{W\left(\sin(\psi_p) + K_h \cos(\psi_p)\right) + V \cos(\psi_p) - T \cos(\psi_T + \psi_p)} \tag{18}$$

where

$$N' = W\left(\cos(\psi_p) - K_h \sin(\psi_p)\right) - U - V \sin(\psi_p) + T \cos(\psi_T + \psi_f) \tag{19}$$

$$W = 0.5\gamma H^2 \left(\left(1 - \left(\frac{Z}{H}\right)^2\right) \cot(\psi_p) - \cot(\psi_f)\right) \tag{20}$$

and

$$\begin{aligned} A &= \frac{(H - z)}{\sin(\psi_p)} \\ U &= 0.5\gamma_w r Z A \\ V &= 0.5\gamma_w r^2 Z^2 \\ r &= \frac{Z_w}{Z}; \end{aligned} \tag{21}$$

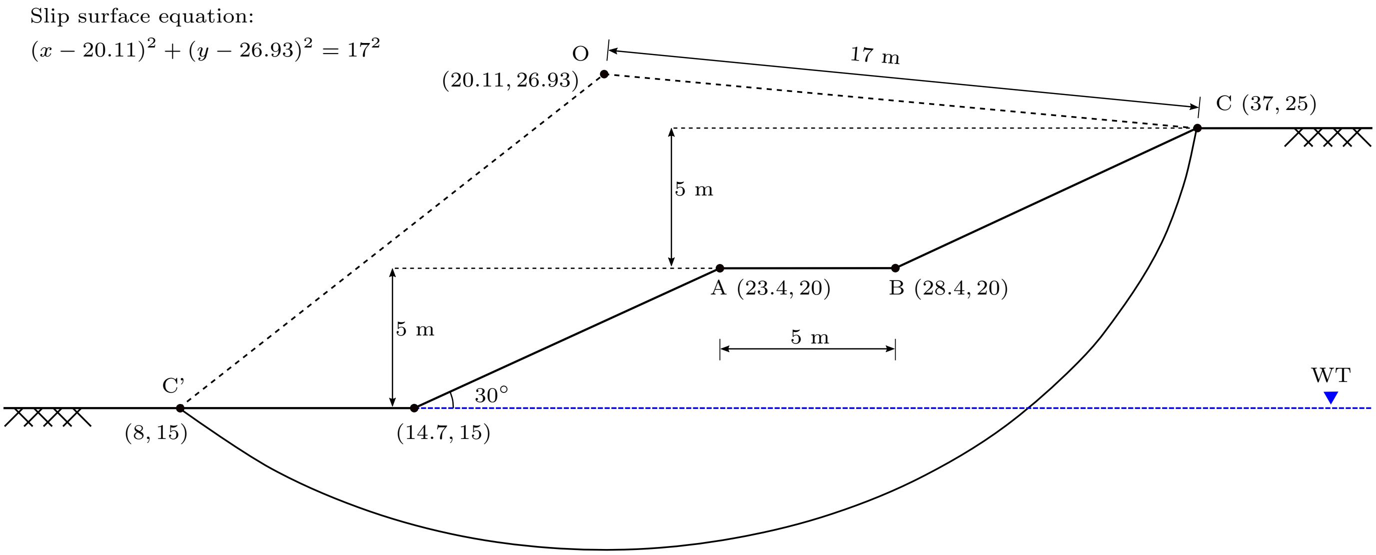

Fig. 3. Homogeneous soil slope with a rotational failure mechanism.

hence the limit state function of the model can formulated as follows:

$$G\left(c, \phi, Z, r, K_h\right) = FS - 1. \tag{22}$$

The input variables c and ϕ are modeled as distributional-free p-boxes based on the CD triaxial tests in Li et al. (2015). Other input variables are listed in table 1.

TABLE 1. Input variables of the plane failure of a rock slope example.

Variable	Modeled as	Units
γ	Trapezoidal fuzzy set - $(23, 25, 27, 29)$	kN/m^3
c	Distributional-free p-box (80% C.I.)	kPa
ϕ	Distributional-free p-box (80% C.I.)	°
z	Three equally credible intervals - $Z_1 = [1, 6]$, $Z_2 = [5, 8]$, and $Z_3 = [7, 10]$	m
r	Triangular PDF - $\mathcal{T}\left(0, 0.33, 1\right)$	–
k_h	Uniform PDF - $\mathcal{U}\left(0.05 - 0.15\right)$	g

The dependence between c and ϕ is modeled with a Frank copula and Kendall's tau of -0.75, resulting in $\mathtt{C}_{Frank}\left(\alpha_1, \alpha_2; -14.138\right)$, where α_1 and α_2 are c and ϕ in the Ω-space. The dependence between Z and r is modeled with a Plackett copula and Spearman's rho of -0.5, leading to $\mathtt{C}_{Plackett}\left(\alpha_3, \alpha_4; 0.195\right)$, where α_3 and α_4 are Z and r. This negative relationship indicates that shallow tension cracks are more likely to be filled with water. γ and K_h are independent of other variables, and c and ϕ are independent of Z and r. The copula relating all variables is:

$$\mathtt{C}\left(\boldsymbol{\alpha}\right) = \mathtt{C}_{Frank}\left(\alpha_1, \alpha_2; -14.138\right) \cdot \mathtt{C}_{Plackett}\left(\alpha_3, \alpha_4; 0.195\right) \cdot \alpha_5 \cdot \alpha_6 \tag{23}$$

here, $\alpha_1, \ldots, \alpha_4$ are previously defined, and α_5 and α_6 represent γ and K_h in the Ω-space, respectively.

The approach in this document allows calculating the bounds of the rock slope's probability of failure (Figure 5), considering the established conditions. The probability bounds are computed for loads $T \in [0, 700]$ kN, in 50 kN intervals.

Results are shown in Figure 6. These results highlight the need for an anchor or other stabilization measures, as the probability of failure bounds are high without an anchor or with low load. As the anchor load increases, both failure probability bounds decrease. As the slope's probability of failure decreases, the gap between the bounds increases, reflecting epistemic uncertainties in the model's basic variables. These uncertainties are more significant when high reliability is needed, as $\underline{P_f}$ and $\overline{P_f}$ can differ by orders of magnitude.

With the proposed approach, designers obtain the tools to, at their discretion, define whether the uncertainties given by the bounds of the probability of failure are acceptable or more information is required to reduce the gap between $\underline{P_f}$ and $\overline{P_f}$. For the model studied in this section, the designers could conclude, for example, that a difference of four orders of magnitude between $\underline{P_f}$ and $\overline{P_f}$ when $T = 600$ kN is unacceptable, and therefore more information is needed to ensure a good reliability on the stability of the rock slope.

Shallow footing with variable width

Consider a square shallow footing resting on cohesive soil. The foundation is situated on horizontal terrain without a groundwater table. The material surrounding the footing follows the Mohr-Coulomb failure criterion. A single concentric vertical load P is applied to the foundation.

It is required to estimate the probability of failure considering

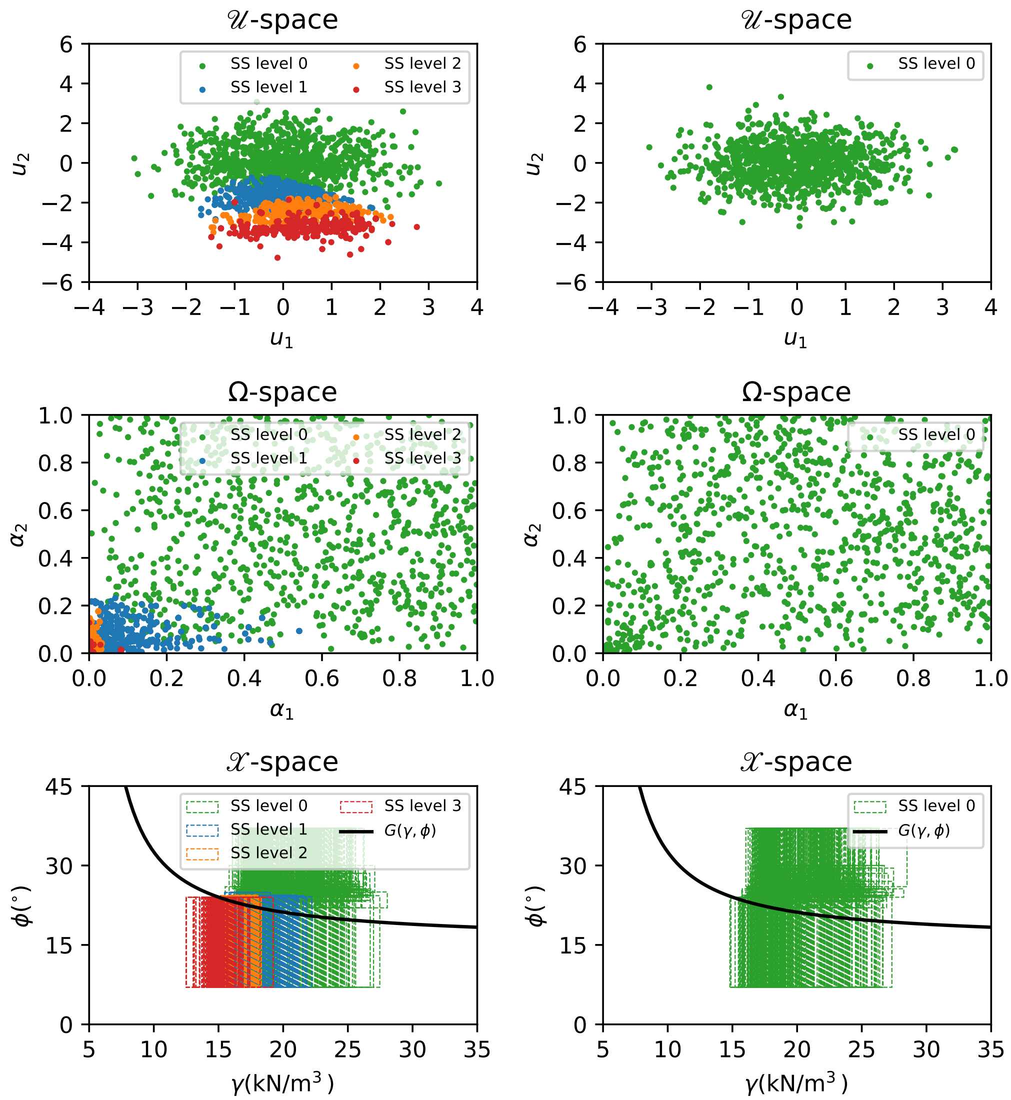

Fig. 4. Evolution of the SubSim samples in the $\mathscr{X}$-space, the Ω-space, and the $\mathscr{U}$-space, for the computation of the lower bound $\underline{P_f}$ (graphs in the left column) and the upper bound $\overline{P_f}$ (graphs in the right column) of the probability of failure of the homogeneous slope example.

that the footing depth D_f is 2.0 meters, and the footing width B varies between 1.0 meter and 2.5 meters. Furthermore, the soil unit weight is a deterministic value set as 20 kN/m^3. Other parameters are treated as random variables. P follows a triangular distribution $\mathcal{T}$ bounded between 800 kN and 2300 kN, with a mode of 1300 kN. Soil cohesion c follows a distributional p-box bounded by the uniform distributions $\mathcal{U}\left([10, 15], [20, 25]\right)$ kN/m^2. The friction angle ϕ is given by a family of three equally-credible intervals $Z_1 = [10, 12]^\circ$, $Z_2 = [11, 13]^\circ$, and $Z_3 = [12, 13]^\circ$. The dependence between c and ϕ is given by a Plackett copula with $\tau = -0.75$, whereas P is an independent variable.

To solve the problem, the equations proposed by Terzaghi (1943) will be used to estimate the ultimate bearing capacity of the foundation, q_c. The system performance function is thus expressed as follows:

$$G\left(c, \phi, B\right) = q_c - \frac{P}{B} \tag{24}$$

such that q_c, for a square shallow footing, is:

$$q_c = 1.3cN_c + \gamma D_f N_q + 0.4\gamma B N_q \tag{25}$$

where N_q, N_c, and N_γ are the Terzaghi's bearing capacity factors (see Terzaghi 1943).

Using the methodology evaluated in this document, the uncertainties in the input parameters are propagated through equation (24). As a result, the bounds for the failure probabilities, as shown in figure 7, are obtained.

In this case, the bounds on the probability of failure are notably narrower compared to the previous examples. This is

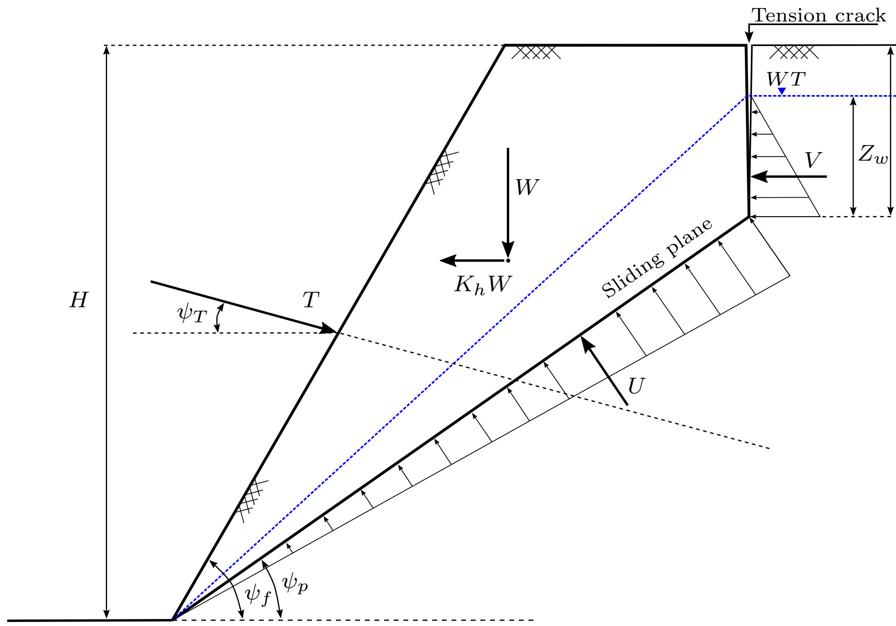

Fig. 5. Two-dimensional stability model of a rock slope with a plane failure mechanism.

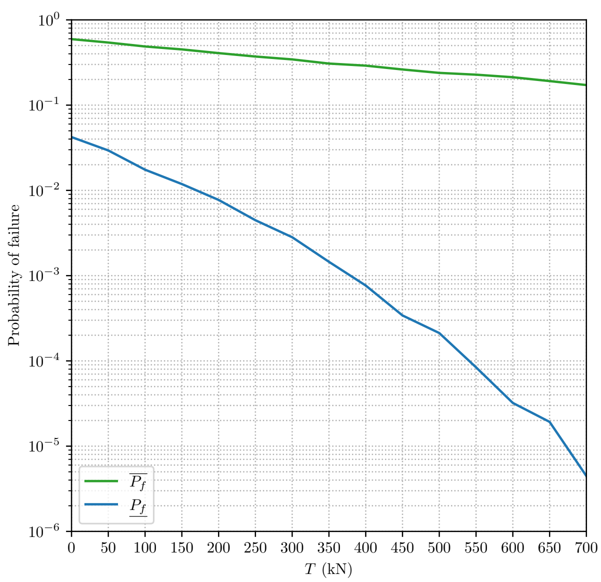

Fig. 6. Evolution on the bounds of the probability of failure in the rock slope from figure 5, considering different load $T \in [0, 700]$ kN transmitted by an anchor to the sliding plane.

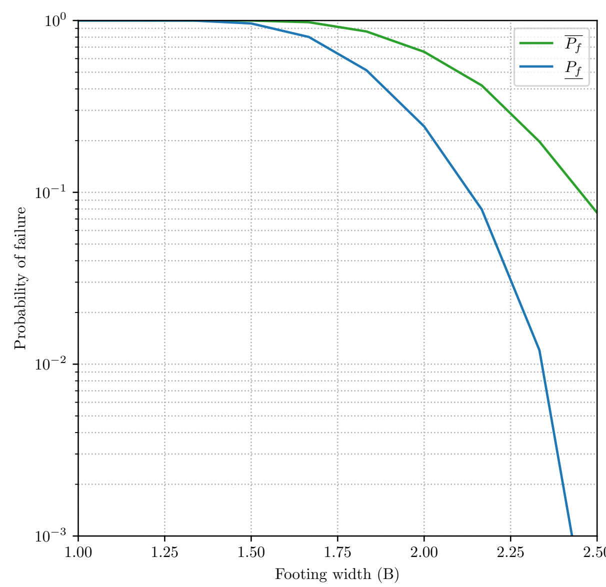

Fig. 7. Evolution on the bounds of the probability of failure in the foundation example, considering different footing widths $B \in [1.0, 2.5]$ m.

mainly because the input variables, such as c and ϕ, exhibit less epistemic uncertainty. As a result, the gap between the upper and lower bounds of the probability of failure is significantly smaller. This demonstrates that the bounds on the probability of failure are highly sensitive to epistemic uncertainty in the input variables. When epistemic uncertainties are smaller, as in this example, the bounds converge more closely. However, when these uncertainties are larger, the gap between the bounds increases significantly, as seen in previous examples.

In this particular example, the difference between the bounds is noticeable only after a foundation width of 1.5 meters. Below this value, the bounds are fairly close, but after 1.5 meters, the separation between the bounds begins to widen more visibly. At the highest footing width, the difference between $\overline{P_f}$ and $\underline{P_f}$ is approximately two orders of magnitude. This is a strong contrast to the previous example, where the difference was as large as five orders of magnitude. This highlights the substantial influence of epistemic uncertainties on the width of the bounds.

This sensitivity is particularly important for geotechnical engineers, as it allows them to assess whether the bounds are adequate for decision-making. Specifically, the upper bound is critical because it represents the worst-case scenario. If this upper bound is too high, it may signal the need for more data or better-quality information to refine the estimate of failure probability. By examining the bounds, engineers can better evaluate the reliability of their models and decide whether the uncertainty is sufficiently accounted for or if additional data collection is necessary to improve confidence in the predictions. Thus, this methodology provides engineers with a clear way to evaluate the adequacy of the probability of failure bounds and to make more informed decisions in their designs.

When several epistemic variables are suspected of producing wide bounds on the probability of failure, a sensitivity analysis is in order, so that the engineer can appropriately identify which variable is the one that requires more attention to reduce the width of the probability of failure interval. In this case, the reader is referred to Patelli et al. (2015), for a method to perform this analysis within the proposed framework.

CONCLUSIONS

This works exposes and evaluates, in the context of geotechnical engineering, the methodology proposed by Alvarez et al. (2018), which integrates subset simulation algorithm, copula theory, and random sets theory, in a comprehensive methodology for conducting reliability analysis.

It is concluded that the evaluated approach is a robust methodology for reliability analysis of geotechnical models, as it accounts for dependencies among the basic variables and incorporates both aleatory and epistemic uncertainties. Furthermore, it is both efficient and accurate, overcoming the major limitation of reliability assessments based solely on probability theory.

The evaluated approach provides upper and lower bounds on the probability of failure of geotechnical models, denoted as $\overline{P_f}$ and $\underline{P_f}$. The gap between these bounds is directly related to epistemic uncertainties — larger epistemic uncertainties lead to a greater gap between these bounds. This gap highlights the risks of neglecting epistemic uncertainty in favor of a single-point probability of failure. Moreover, it is found that epistemic uncertainties have a greater impact when expected probabilities of failure are small. For example, in the second case study in Section 6, the gap between $\overline{P_f}$ and $\underline{P_f}$ increases significantly as the probability of failure decreases.

Neglecting epistemic uncertainties, as is commonly done in many traditional analyses, can be dangerous. It may lead to overly optimistic estimates of the failure probability or, conversely, to overly conservative assessments. This underscores the importance of explicitly considering epistemic uncertainties, which, as shown here through three different examples, can significantly alter the resulting probability bounds and affect the overall reliability analysis.

It is important to note that the evaluated methodology does not aim to invalidate the use of safety factors in geotechnical design. On the contrary, it provides a comprehensive understanding of the uncertainties behind each model, which can complement the use of safety factors. With this methodology, geotechnical engineers can assess whether the probability of failure in their models is acceptable or if further actions are needed to reduce uncertainty and improve reliability. Additionally, the bounds on the probability of failure obtained from this methodology can be used for further analysis, such as evaluating loss distributions, assessing value at risk, or computing expected utilities. These extensions provide engineers with a deeper insight into the potential risks associated with their designs, enabling more informed decision-making in the face of uncertainty. The ability to quantify both the upper and lower bounds of failure probability allows for more robust risk management strategies, making it a valuable tool in the broader context of geotechnical risk analysis and decision theory.

We acknowledge that further research is needed to extend our methodology to more complex geotechnical analyses, such as those considering random fields of finite element methods. Our approach suffers from the curse of dimensionality with the number of epistemic variables, and addressing this challenge will require the development of more efficient methodologies, which we consider an important direction for future research.

Additionally, we recognize that there are other methods to address epistemic uncertainties, such as Bayesian updating. A key distinction between our methodology and Bayesian approaches is in how epistemic uncertainty is treated. While Bayesian methods integrate over uncertain parameters to yield a single probability of failure, our approach — using random set theory — explicitly accounts for both aleatory and epistemic uncertainties, providing bounds on failure probability rather than a single value. These bounds enable engineers to evaluate whether the available data is sufficient or if additional data collection is needed. Furthermore, our methodology efficiently handles high-dimensional dependence structures using copulas and maintains computational feasibility through subset simulation.

Importantly, the methodology evaluated in this research makes no assumptions about the available data and can work directly with the information provided. For example, it can handle intervals for random variables (e.g., c and ϕ) provided by experienced geotechnical engineers, without requiring specific prior distributions. In contrast, Bayesian approaches typically rely on prior distributions, often assuming a uniform distribution when no prior information is available, based on the principle of indifference.

Data Availability Statement

All the codes used for the examples presented in this paper, as well as additional resources, are publicly available at the following repository: `https://github.com/jjsepulvedag/Pf_Bounds-GeoEng`.

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